Mathematics
PhD Prelim Exam Spring 2020

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1. **(30pts)** Consider the following linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

Determine $\boldsymbol{x}(t)$ when

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix};$$

please express your final answer entirely in terms of real-valued quantities to facilitate its interpretation.

2. **(50pts)** Consider a rectangular membrane, occupying the region $0 \le x \le 4$ and $0 \le y \le 6$, whose deflection is governed by

$$\ddot{u} = c^2 \nabla^2 u\,.$$

The membrane is fixed on its perimeter. Using separation of variables, determine the frequency of the second natural mode of vibration. Assume $c = 1/\pi$.

3. **(20 pts)** Evaluate the surface integral

$$I = \int_{\mathcal{S}} (x_1 n_1 + 8.0 x_2 n_2 + 11.0 x_3 n_3)\, dA\,,$$

where $\mathcal{S} = \mathcal{C} \cup \mathcal{D}$ consisting of the cylinder

$$\mathcal{C} = \{(x_1, x_2, x_3) \mid (x_1)^2 + (x_2)^2 = a^2 \text{ and } 0 \le x_3 \le b)\}$$

and the circular disks

$$\mathcal{D} = \left\{(x_1, x_2, x_3) \mid (x_1)^2 + (x_2)^2 \le a^2 \text{ and } x_3 \in \{0, b\}\right\}.$$

Note a and b are given constants, and $\boldsymbol{n}$ is the outward unit normal to the enclosed volume.

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PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1 (50 points)
Considering an exponential distribution (function) model,

$$f(t) = \frac{1}{\lambda}\exp(-t/\lambda),$$

with an unknown parameter λ.

We have collected a set of data $\{t_i\}_{i=1}^{10}$ in measurements that satisfy the distribution function model with the unknown parameter,

$$\{t_i\}_{i=1}^{10} = \{1.2, 3.0, 6.3, 10.1, 5.2, 2.4, 7.1, 6.9, 4.2, 3.8\} \ .$$

To quantify the unknown parameter λ with the data, we require that the unknown parameter λ making the following objective function maximum, i.e. there exists a parameter $\hat{\lambda}$ such that it makes $\log L(\hat{\lambda}) = \max_{\lambda \in \mathbb{R}}(\log L(\lambda))$, where

$$L(\lambda) = \Pi_{i=1}^{10}\frac{1}{\lambda}\exp(-t_i/\lambda) = \frac{1}{\lambda}\exp(-t_1/\lambda)\cdot\frac{1}{\lambda}\exp(-t_2/\lambda)\cdot \cdots \cdot \frac{1}{\lambda}\exp(-t_{10}/\lambda) \ .$$

This is usually expressed as

$$\hat{\lambda} = \arg\ \max_{\lambda \in \mathbb{R}}\ \log(L(\lambda)) \ .$$

(a) Find $\hat{\lambda}$?
(b) How do I know that $\hat{\lambda}$ maximizes $\log L(\lambda)$?

Problem 2 (50 points)
Consider the following 3×3 matrix,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(1) Find all the eigenvalues of the matrix $[\mathbf{A}]$;
(2) Find all the corresponding eigenvectors ;
(3) Find the eigenvalues of the matrix $\mathbf{B} = \mathbf{A}^3$.

UNIVERSITY OF CALIFORNIA, BERKELEY College of Engineering

Department of Civil and Environmental Engineering
Structure Engineering, Materials, and Mechanics (SEMM)

2019 Spring PhD Preliminary Exam: Mathematics

Problem 1. (60 Points)

Consider the following cubic polynomial interpolation function for a beam element,

$$v(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3, \quad \forall\ \xi \in (0, L)$$

and let that

$$v(\xi) = N_1(\xi)v_0 + N_2(\xi)\theta_0 + N_3(\xi)v_L + N_4(\xi)\theta_L = \sum_{I=1}^{4} N_I(\xi)u_I$$

where $\xi \in (0, L)$, and $u_1 = v_0; u_2 = \theta_0; u_3 = v_L$, and $u_4 = \theta_L$.
Thus, we have

$$\begin{aligned} v(0) &= c_0 \\ \theta(0) &= c_1 \\ v(L) &= c_0 + c_1 L + c_2 L^2 + c_3 L^3 \\ \theta(L) &= c_1 + 2c_2 L + 3c_3 L^2 \end{aligned}$$

Find c_0, c_1, c_2 and c_3 by solving the following equation,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v(0) \\ \theta(0) \\ v(L) \\ \theta(L) \end{bmatrix}$$

and identify

$$N_1(\xi), N_2(\xi), N_3(\xi), \text{ and } N_4(\xi)\ .$$

Problem 2. (40 Points)
Let

$$\begin{aligned} H_1(\xi) &= \Big(1 - 3\Big(\frac{\xi}{L}\Big)^2 + 2\Big(\frac{\xi}{L}\Big)^3\Big) \quad \text{and} \\ H_2(\xi) &= \xi\Big(1 - \frac{\xi}{L}\Big)^2 . \end{aligned}$$

Calculate the following elements of stiffness:

$$[K_{ij}] = \int_0^L H_i''(\xi) EI H_j''(\xi) d\xi \ , \quad i, j = 1, 2$$

where EI are constant, and $H_i''(\xi) := \frac{d^2 H_i}{d\xi^2}, i = 1, 2.$

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Problem 1. (40 points)

Define

$$G(t) = \int_0^\infty \Big(\int_{-\infty}^{\frac{t}{\sqrt{\nu}}\sqrt{u}} \frac{(1/2)^{\nu/2}}{\sqrt{2\pi}\Gamma(\nu/2)} u^{\nu/2-1} \exp(-\frac{z^2+u}{2}) dz \Big) du \tag{1}$$

where ν is a constant, and $\Gamma(\nu/2)$ is Gamma function, which you do not need to evaluate. Just leave it there.

Calculate or find the expression for

$$g(t) = \frac{dG(t)}{dt} \ ?$$

Hint: Apply the fundamental theorem of calculus and chain rule. You do not need to integrate u.

Problem 2 (60 points)

Consider a smooth function $f(x) \geq 0$, and

$$\frac{df}{dx} = -\frac{f^2(x)}{1-F(x)} =: f'(x)$$

where $1 \geq F(x) = \int_{-\infty}^x f(t)dt > 0$ or $F'(x) = \dfrac{dF}{dx} = f(x)$.

(1) Calculate $f''(x)$, $f'''(x) \cdots$ and verify that

$$\frac{d^k f}{dx^k}(x) = (-1)^k \frac{f^{k+1}(x)}{(1-F(x))^k}, \quad k = 1, 2, \cdots$$

i.e. assume that

$$\frac{d^k f}{dx^k}(x) = (-1)^k \frac{f^{k+1}(x)}{(1-F(x))^k}, \quad k = 1, 2, \cdots$$

show that

$$\frac{d^{k+1} f}{dx^{k+1}}(x) = (-1)^k \frac{f^{k+2}(x)}{(1-F(x))^{k+1}}, \quad k = 1, 2, \cdots$$

(2) Let

$$F_n(x) := [F(x)]^n \ , \ n = 1, 2, \cdots$$

Find

$$f_n(x) := \frac{dF_n}{dx} = ?$$

and show that

$$f_n'(x) = \frac{d^2 F_n}{dx^2} = n f'(x)[F(x)]^{n-1} + n(n-1) f^2(x)[F(x)]^{n-2}$$

(3) Assume that at $x = x_n$, $f_n'(x_n) = 0$, find

$$F(x_n) = \ ?$$

(4) Assume that

$$\frac{f(x_n)}{(1 - F(x_n)} = \alpha_n = const. \quad \rightarrow \quad \text{Find } f(x_n) \ ?$$

(5) Consider the Taylor series expansion of $F(x)$ at $x = x_n$, i.e.

$$F(x) = F(x_n) + F'(x)(x - x_n) + \frac{1}{2!} F''(x_n)(x - x_n)^2 + \frac{1}{3!} F'''(x_n)(x - x_n)^3 + \cdots$$

Show that

$$F(x) = 1 - \frac{1}{n}\Big[1 - \frac{\alpha_n (x - x_n)}{1!} + \frac{\alpha_n^2 (x - x_n)^2}{2!} - \frac{\alpha_n^3 (x - x_n)^3}{3!} + \cdots\Big]$$

and subsequently

$$F(x) = 1 - \frac{1}{n} \exp(-\alpha_n (x - x_n)) \ .$$

UNIVERSITY OF CALIFORNIA, BERKELEY Dept. of Civil and Environmental Engineering
SPRING SEMESTER 2018 Structural Engineering, Mechanics and Materials

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PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1. (40 points)

Define an average operator in $\mathbf{R}$ as

$$< f > (x) := \int_{-\infty}^{\infty} f(y) \exp(-a(x-y)^2) dy, \text{ where } |f(y)| < c, \forall y \in \mathbf{R}$$

where $a > 0$ and $0 < c < \infty$ are real numbers.
Show that

$$\frac{d}{dx} < f > (x) = < \frac{df}{dy} > \quad . \tag{1}$$

Problem 2. (40 points)

Consider the following differential equation,

$$EI \frac{d^4 v}{dx^4} = q(x), \quad \forall\ 0 < x < L \tag{2}$$

where $g(x)$ is a given function, and the differential equation has the following boundary conditions:

$$v(0) = 0, \quad v'(0) = 0, \quad EIv''(L) = \bar{M}, \ EIv'''(L) = \bar{V}$$

Consider a given function $w(x)$ with the boundary conditions

$$w(0) = 0, \quad w'(0) = 0\ , \quad w(L) = 1, \quad w'(L) = -1.$$

Evaluate the following definite integral,

$$\int_0^L EIv''(x)w''(x)dx = ?$$

where $v' = \frac{dv}{dx}$, $v'' = \frac{d^2v}{dx^2}$ and $v''' = \frac{d^3v}{dx^3}$.

Problem 3. (20 points)

Consider the following algebraic equation,

$$\begin{bmatrix} 1 \\ x \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ x_1 & x_2 \end{bmatrix} \begin{bmatrix} N_1(x) \\ N_2(x) \end{bmatrix}$$

where $N_1(x)$ and $N_2(x)$ are unknown functions, and x_1, x_2 are two given points in the real number axis $\mathbf{R}$. Find $N_1(x)$ and $N_2(x)$?
Under which condition, $N_1(x)$ and $N_2(x)$ do not exit.