Mathematics

Problem 1 (30 Points)

Consider a 4×4 matrix defined in the following,

$$A = \begin{bmatrix} 0 & \ln x & 0 & \ln x \\ 0 & 0 & 1 & 0 \\ x & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} ,$$

where $x > 0$. Let matrix B be defined as

$$B = A^{2021} .$$

Find

$$\frac{d}{dx} \det(B) .$$

Hint: for square matrices C and D of equal size, the determinant of their matrix product equals the product of their determinants, i.e., $\det(CD) = \det(C) \det(D)$.

Problem 2 (40 Points)

Consider a n-dimensional vector $\boldsymbol{x} = [x_1, ..., x_n]$. Vector $\boldsymbol{x}$ is a "probability vector" such that $\sum_{i=1}^{n} x_i = 1$, and $\forall i$, $i = 1, ..., n$, $x_i \geq 0$. Define the entropy as

$$\mathcal{H}(\boldsymbol{x}) = -\sum_{i=1}^{n} x_i \ln x_i ,$$

where by definition $0 \ln 0 \equiv 0$.

(a) Find the vector x^* that maximizes the entropy.

(b) Solve question (a) again, subjected to an additional linear constraint: $\sum_{i=1}^{n} \epsilon_i x_i = E$, $\epsilon_i > 0$, where ϵ_i and E are fixed/known.

Note: i) You do not need to know anything about the theory of probability to solve this problem; ii) For question (b) you do not need to find the analytical solution of x^*.

Problem 3 (30 Points)

A popular model for the growth of a population is written in the following form

$$\frac{dN}{dt} = \lambda N \left(1 - \frac{N}{K}\right) , N(0) = N_0 , N_0 \geq 0 ,$$

where $\lambda > 0$ is the intrinsic growth rate of the population, and K, $K > N_0$, is the carrying capacity of the environment. Solve this equation, and make a qualitative plot of $N(t)$.

Mathematics
PhD Prelim Exam Spring 2021

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1. **(30pts)** Consider an initial value problem governed by the ordinary differential equation

$$\ddot{y} - 5\dot{y} + 6y = e^{2t}$$

with initial conditions $y(0) = 1$, $\dot{y}(0) = 1$. Solve for $y(t)$.

2. **(40 pts)** Consider a partial differential equation for $y(x, t)$

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{\lambda}\frac{\partial^2 y}{\partial t^2}$$

where $x \in [0, l]$, $t \in [0, \infty)$, and $\lambda, l > 0$ are given. Find $y(x, t)$ assuming that $\dot{y}(x, t)|_{t=0} = 0$, and

$$y(x, 0) = \begin{cases} \frac{2hx}{l} & x \leq l/2 \\ 2h - \frac{2hx}{l} & x > l/2\,, \end{cases}$$

where $h > 0$ is given.

3. **(30pts)** Consider three vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^4$, where

$$\begin{aligned} \boldsymbol{a} &= (1, 2, 0, 3)^T \\ \boldsymbol{b} &= (0, 1, 2, 3)^T \\ \boldsymbol{c} &= (1, 1, 2, 2)^T\,. \end{aligned}$$

Find an orthonormal basis for

$$\text{span}\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}\}$$

using the Gram-Schmidt method.

Mathematics
PhD Prelim Exam Spring 2020

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1. **(30pts)** Consider the following linear system $\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}$, where

$$\boldsymbol{A} = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}.$$

Determine $\boldsymbol{x}(t)$ when

$$\boldsymbol{x}(0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix};$$

please express your final answer entirely in terms of real-valued quantities to facilitate its interpretation.

2. **(50pts)** Consider a rectangular membrane, occupying the region $0 \leq x \leq 4$ and $0 \leq y \leq 6$, whose deflection is governed by

$$\ddot{u} = c^2 \nabla^2 u\,.$$

The membrane is fixed on its perimeter. Using separation of variables, determine the frequency of the second natural mode of vibration. Assume $c = 1/\pi$.

3. **(20 pts)** Evaluate the surface integral

$$I = \int_{\mathcal{S}} (x_1 n_1 + 8.0 x_2 n_2 + 11.0 x_3 n_3)\; dA\,,$$

where $\mathcal{S} = \mathcal{C} \cup \mathcal{D}$ consisting of the cylinder

$$\mathcal{C} = \{(x_1, x_2, x_3) \mid (x_1)^2 + (x_2)^2 = a^2 \text{ and } 0 \leq x_3 \leq b)\}$$

and the circular disks

$$\mathcal{D} = \left\{(x_1, x_2, x_3) \mid (x_1)^2 + (x_2)^2 \leq a^2 \text{ and } x_3 \in \{0, b\}\right\}.$$

Note a and b are given constants, and $\boldsymbol{n}$ is the outward unit normal to the enclosed volume.

NAME ____________________________

PH.D. PRELIMINARY EXAMINATION

MATHEMATICS

Problem 1 (50 points)
Considering an exponential distribution (function) model,

$$f(t) = \frac{1}{\lambda}\exp(-t/\lambda),$$

with an unknown parameter λ.

We have collected a set of data $\{t_i\}_{i=1}^{10}$ in measurements that satisfy the distribution function model with the unknown parameter,

$$\{t_i\}_{i=1}^{10} = \{1.2, 3.0, 6.3, 10.1, 5.2, 2.4, 7.1, 6.9, 4.2, 3.8\} \ .$$

To quantify the unknown parameter λ with the data, we require that the unknown parameter λ making the following objective function maximum, i.e. there exists a parameter $\hat{\lambda}$ such that it makes $\log L(\hat{\lambda}) = \max_{\lambda \in \mathbb{R}}(\log L(\lambda))$, where

$$L(\lambda) = \Pi_{i=1}^{10}\frac{1}{\lambda}\exp(-t_i/\lambda) = \frac{1}{\lambda}\exp(-t_1/\lambda)\cdot\frac{1}{\lambda}\exp(-t_2/\lambda)\cdot\cdots\cdot\frac{1}{\lambda}\exp(-t_{10}/\lambda) \ .$$

This is usually expressed as

$$\hat{\lambda} = \arg\ \max_{\lambda\in\mathbb{R}}\ \log(L(\lambda)) \ .$$

(a) Find $\hat{\lambda}$?
(b) How do I know that $\hat{\lambda}$ maximizes $\log L(\lambda)$?

Problem 2 (50 points)
Consider the following 3×3 matrix,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

(1) Find all the eigenvalues of the matrix $[\mathbf{A}]$;
(2) Find all the corresponding eigenvectors ;
(3) Find the eigenvalues of the matrix $\mathbf{B} = \mathbf{A}^3$.

UNIVERSITY OF CALIFORNIA, BERKELEY College of Engineering

Department of Civil and Environmental Engineering
Structure Engineering, Materials, and Mechanics (SEMM)

2019 Spring PhD Preliminary Exam: Mathematics

Problem 1. (60 Points)

Consider the following cubic polynomial interpolation function for a beam element,

$$v(\xi) = c_0 + c_1\xi + c_2\xi^2 + c_3\xi^3, \quad \forall \ \xi \in (0, L)$$

and let that

$$v(\xi) = N_1(\xi)v_0 + N_2(\xi)\theta_0 + N_3(\xi)v_L + N_4(\xi)\theta_L = \sum_{I=1}^{4} N_I(\xi)u_I$$

where $\xi \in (0, L)$, and $u_1 = v_0; u_2 = \theta_0; u_3 = v_L$, and $u_4 = \theta_L$.
Thus, we have

$$\begin{aligned} v(0) &= c_0 \\ \theta(0) &= c_1 \\ v(L) &= c_0 + c_1 L + c_2 L^2 + c_3 L^3 \\ \theta(L) &= c_1 + 2c_2 L + 3c_3 L^2 \end{aligned}$$

Find c_0, c_1, c_2 and c_3 by solving the following equation,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} v(0) \\ \theta(0) \\ v(L) \\ \theta(L) \end{bmatrix}$$

and identify

$$N_1(\xi), N_2(\xi), N_3(\xi), \text{ and } N_4(\xi) \ .$$

Problem 2. (40 Points)
Let

$$\begin{aligned} H_1(\xi) &= \Big(1 - 3\Big(\frac{\xi}{L}\Big)^2 + 2\Big(\frac{\xi}{L}\Big)^3\Big) \quad \text{and} \\ H_2(\xi) &= \xi\Big(1 - \frac{\xi}{L}\Big)^2 . \end{aligned}$$

Calculate the following elements of stiffness:

$$[K_{ij}] = \int_0^L H_i''(\xi) EI H_j''(\xi) d\xi \ , \quad i, j = 1, 2$$

where EI are constant, and $H_i''(\xi) := \frac{d^2 H_i}{d\xi^2}, i = 1, 2.$