Mechanics
PhD Prelim Exam Fall 2021

All questions have equal weighting.

1. Assume a body $\mathcal{B}$ and a force system $[\boldsymbol{b}(\boldsymbol{x}), \boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n})]$ composed of body-forces and surface tractions on surfaces with normal $\boldsymbol{n}$, at all points $\boldsymbol{x} \in \mathcal{B}$. Assuming only global linear momentum balance, show that $\boldsymbol{t}(\boldsymbol{x}, \boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, -\boldsymbol{n})$. Do not assume Cauchy's law to be true as this result is needed to prove Cauchy's law.

2. Consider an isotropic linear elastic body with constitutive relation

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \, ,$$

where Latin subscripts take on values 1, 2, and 3; Einstein summation convention is assumed throughout this problem. Assume the body to be in a state of plane strain (in the 1-2 plane) and derive the plane strain constitutive law

$$\epsilon_{\alpha\beta} = A\sigma_{\alpha\beta} - B\sigma_{\gamma\gamma}\delta_{\alpha\beta} \, ,$$

where Greek subscripts only take on the values 1 and 2. Make sure to define A and B in terms of E and ν.

3. Consider a linear viscoelastic material with a (one-dimensional) power law relaxation function/modulus

$$E_r(t) = E_1 + E_2\, t^{-p} \, ,$$

where the constants $E_1, E_2 > 0$ and $1 < p$.

   (a) What is the glassy modulus E_{rg} for this material? What is the equilibrium modulus E_{re} for this material.

   (b) Write an expression for the stress as a function of time, $\sigma(t)$, valid for any given strain history, $\epsilon(t)$.

   (c) Assume now that

$$\epsilon(t) = \begin{cases} 0 & t \leq 0 \\ \dot{\epsilon}_o\, t & t \geq 0 \, , \end{cases}$$

   where $\dot{\epsilon}_o$ is a given constant. Find and sketch the stress as a function of time.

UNIVERSITY OF CALIFORNIA AT BERKELEY
SPRING SEMESTER 2021

Department of Civil and Environmental Engineering
Structural Engineering, Mechanics and Materials

Name ________________________________

Doctoral Preliminary Examination

Mechanics

Problem 1. (40 points)
Consider that the stress tensor can be decomposed into the sum of hydrostatic and deviatoric stresses as follows

$$\sigma_{ij} = \frac{1}{3}\delta_{ij}\sigma_{kk} + \sigma'_{ij}$$

where

$$\frac{1}{3}\delta_{ij}\sigma_{kk}$$

is the hydrostatic term and σ'_{ij} is the deviatoric stress. The same is true for strain,

$$\epsilon_{ij} = \frac{1}{3}\delta_{ij}\epsilon_{kk} + \epsilon'_{ij}$$

(1) Express the elastic energy density

$$W = \frac{1}{2}\boldsymbol{\sigma} : \boldsymbol{\epsilon} \tag{1}$$

in terms of hydrostatic stress and strain and deviatoric stress and strain.

(2) Let

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & \sigma/2 \\ 0 & \sigma/2 & 0 \end{bmatrix}$$

Calculate $\boldsymbol{\sigma}'$ and $\sigma_{VM} := \sqrt{\frac{3}{2}\boldsymbol{\sigma}' : \boldsymbol{\sigma}'}$.

Problem 2. (40 points)

Let a material point with the undeformed coordinates (X_1, X_2, X_3) and the deformed coordinates (x_1, x_2, x_3). Assume that the deformation map can be expressed explicitly as

$$x_1 = (1+a)X_1 + bX_2 + cX_3; \tag{2}$$
$$x_2 = -bX_1 + (1+a)X_2 + dX_3 \tag{3}$$
$$x_3 = -cX_1 - dX_2 + (1+a)X_3 \tag{4}$$

For a material line segment $\mathbf{X}_{OP} = \mathbf{X}_P - \mathbf{X}_O$ where

$$\mathbf{X}_O = (0, 0, 0), \quad \mathbf{X}_P = (\Delta, \Delta, \Delta),$$

where Δ is a constant parameter.
(a) Find $\mathbf{x}_{OP} = \mathbf{x}_P - \mathbf{x}_O$?
(b) Find the elongation of $\mathbf{X}_{OP}$?
(c) Find the deformation gradient $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$?
(c) Find the Lagrangian strain

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{I}) \; ?$$

Problem 3. (20 points)

Let

$$\mathbf{x}(t) = \mathbf{X} + \mathbf{v} t \; .$$

where $\mathbf{v}$ is a fixed vector.

Consider a vector function

$$\mathbf{z}(t) = \mathbf{z}(\mathbf{x}(t), t)$$

Calculate material time derivative

$$\frac{D}{Dt}\mathbf{z}(t) = ?$$

Mechanics
PhD Prelim Exam Spring 2020

All questions have equal weighting.

1. Consider single crystal composed of cubic Manganese, `Mn`. In a basis $\{\boldsymbol{e}_i\}$ aligned with the crystal's cubic axes the compliance components in Voigt notation are

$$\mathbb{S} = \begin{bmatrix} 5.6 & -2.2 & -2.2 & 0 & 0 & 0 \\ -2.2 & 5.6 & -2.2 & 0 & 0 & 0 \\ -2.2 & -2.2 & 5.6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10.1 \end{bmatrix} [1/\text{TPa}]\,.$$

   What is the Young's modulus for the material in the direction $\boldsymbol{d} = (\boldsymbol{e}_1 + \boldsymbol{e}_2)/\sqrt{2}$?

2. At a given point $\boldsymbol{x}_o$ in a body $\mathcal{B}$, the traction is known to be

$$\boldsymbol{t} = 1\boldsymbol{e}_1 + 2\boldsymbol{e}_2 \text{ MPa}$$

   on the plane with normal $\boldsymbol{n} = \boldsymbol{e}_1$; it is known to be

$$\boldsymbol{t} = 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3 \text{ MPa}$$

   on the plane with normal $\boldsymbol{n} = \boldsymbol{e}_3$; and it is known to be

$$\boldsymbol{t} = 2/\sqrt{2}\boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 5/\sqrt{2}\boldsymbol{e}_3 \text{ MPa}$$

   on the plane with normal $\boldsymbol{n} = (\boldsymbol{e}_2 + \boldsymbol{e}_3)/\sqrt{2}$. What is the state of stress at $\boldsymbol{x}_o$

3. Starting from the strong form of the equilibrium equations and assuming small strain, show that power of the external loads on a body $\mathcal{B}$

$$\mathcal{P}_{\text{ext}} = \int_{\partial\mathcal{B}} t_i v_i \, dA + \int_{\mathcal{B}} b_i v_i \, dV$$

   is equal to the stress power plus the rate of change of the kinetic energy

$$\mathcal{P}_{\text{int}} = \int_{\mathcal{B}} \left(\dot{\epsilon}_{ij}\sigma_{ij} + \dot{\overline{\tfrac{1}{2}\rho v^2}} \right) dV\,,$$

   where v_i are the velocity components, ϵ_{ij} are the small strain components, σ_{ij} are the stress components, ρ is the mass density, t_i are the traction components, and b_i are the volumetric body force components (force/volume). You must justify each step of your derivation with a short statement.

Mechanics
PhD Preliminary Fall 2019

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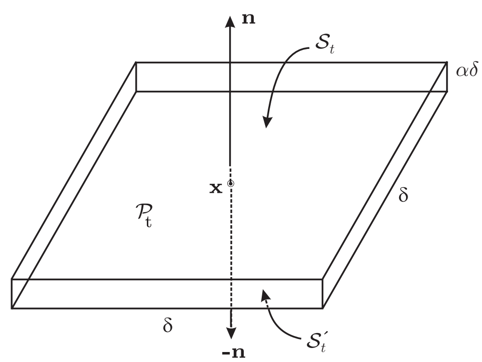
1. Linear elastic monoclinic materials have a single plane of reflective symmetry. Show that such materials have only 13 (independent) non-zero elastic moduli.

2. Assuming only global linear and angular momentum balance, show using a *pill-box* construction, that the traction vector on a surface with normal $\boldsymbol{n}$ at any point in a body $\mathcal{B}$ is an odd function of the surface normal, i.e. show

$$\boldsymbol{t}(\boldsymbol{x},\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x},-\boldsymbol{n}) \qquad \forall \boldsymbol{x} \in \mathcal{B}\,, \quad \forall \boldsymbol{n}\,. \tag{1}$$



3. Consider a bolt of diameter D and assume that the bolt is made from a linear viscoelastic material with *given* stress relaxation modulus $E_r(t)$. The bolt is used to join two *rigid* plates. An impact drill is used to tighten the bolt very quickly and the instantaneous force in the bolt at time $t = 0^+$ is known to be P_o.

   (a) Find an expression for the strain in the bolt in terms of P_o and $E_r(\cdot)$.

   (b) Find an expression for the force in the bolt at any time $t > 0$ in terms of $E_r(\cdot)$ and your answer to Part 3a.

   Treat this problem as one-dimensional. Ignore all three-dimensional aspects of the problem.