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**Key Points:**

- Develops a model for analytic soil moisture PDFs and crossing times in seasonally dry regions
- Presents a novel representation of soil moisture conditions at the beginning of the dry season
- Successful test using AmeriFlux soil moisture data from Tonzi Ranch

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# A minimal probabilistic model for soil moisture in seasonally dry climates

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**Abstract** In seasonally dry climates, a distinct rainy season is followed by a pronounced dry season during which rainfall often makes a negligible contribution to soil moisture. Using stochastic analytical models of soil moisture to represent the effects of this seasonal change has been hindered by the need to mathematically represent the stochastic influence of wet season climate on dry season soil water dynamics. This study presents a simple process-based stochastic model for soil moisture dynamics, which explicitly models interseasonal transient dynamics while accounting for carry over soil moisture storage between the wet and dry seasons, and allows a derivation of an analytical expression for the dry season mean first passage time below a soil moisture threshold. Such crossing times pose controls on both vegetation productivity and water stress during dry summers. The new model, along with an existing model that incorporates nonzero dry season rainfall but not variability in the soil moisture condition at the start of the dry season, are tested against data from the Tonzi Ranch Ameriflux site. Both models predict first passage times well for high soil moisture thresholds, but the new model improves prediction at lower thresholds. The annual soil moisture probability distribution function (PDF) from the new model also compares well with observations.

## 1. Introduction

Seasonally dry ecosystems (SDEs), which include Mediterranean, tropical monsoonal, and tropical savannah climates, cover approximately 30% of the Earth's land area [*Peel et al.*, 2007] and contain several biodiversity hot spots [*Miles et al.*, 2006; *Klausmeyer and Shaw*, 2009]. Pronounced climatic variability is a common feature of these regions [*Fatichi et al.*, 2012] and is projected to intensify in future climate scenarios [*Gao and Giorgi*, 2008; *García-Ruiz et al.*, 2011; *Dominguez et al.*, 2012]. Consequently, a number of studies classify SDEs and their water resources as climatically vulnerable [*Nohara et al.*, 2006; *Parry*, 2007; *Gao and Giorgi*, 2008; *Klausmeyer and Shaw*, 2009; *García-Ruiz et al.*, 2011]. Projecting the variability of water availability and the risks of water shortfalls in these regions could therefore provide useful insights into vegetation and ecosystem risk [*Vico et al.*, 2015; *Müller et al.*, 2014].

Process-based stochastic methods provide a minimal modeling framework to obtain the probability distributions of soil moisture and streamflow [*Milly*, 1993; *Rodríguez-Iturbe et al.*, 1999; *Laio et al.*, 2002; *Botter et al.*, 2007]. Since hydroclimatic variation strongly impacts plants through soil moisture [*Taiz and Zeiger*, 2010; *Thompson and Katul*, 2012], these models have also been used to predict ecological response and to assess the vulnerability of ecosystems [*Porporato et al.*, 2004; *Viola et al.*, 2008; *Thompson et al.*, *2013* 2014]. To date, the majority of methods have been developed under conditions where either the climatic forcing can be considered stationary in time [*Rodríguez-Iturbe et al.*, 1999; *Porporato et al.*, 2004; *Botter et al.*, 2007] or where transient dynamics between seasons are not considered [*Miller et al.*, 2007; *Kumagai et al.*, 2009]. Studies that considered the effects on soil moisture of seasonality in rainfall or evaporative demand, or transient dynamics between seasons, have typically focused on the mean soil moisture dynamics [*D'Odorico et al.*, 2000; *Laio et al.*, 2001; *Feng et al.*, 2012; *Feng et al.*, 2015].

*Viola et al.* [2008] first investigated the impacts of transient soil moisture dynamics on plant water stress during the growing season in Mediterranean ecosystems. In that study, the steady state PDF of soil moisture during the wet season represents the end of wet season conditions, after which the dry season proceeds following a step change in rainfall statistics and potential evapotranspiration. For a site with shallow soil or a small mean rainfall depth (relative to the total possible amount of soil water storage), this approach is appropriate because the variance of wet season conditions will be small. However, to accurately quantify soil moisture variability at sites with large average rainfall depths or soil storage, a more precise description of the conditions at the start of the dry season may be required.

This work developed a simplified but fully analytic stochastic theory for soil moisture probability distribution functions for seasonally dry regions. The model yields a single, analytical expression for the annually integrated soil moisture PDF under seasonal climates, and a similarly minimal analytical expression for the mean first passage time of dry season soil moisture below a given threshold. Such probabilistic descriptions of first passage times are used to link soil moisture dynamics to plant water stress [*Rodríguez-Iturbe et al*., 2001; *Viola et al*., 2008].

The model of *Porporato et al*. [2004] was used to represent the probabilistic dynamics of soil moisture during the wet season. In this framework, the wet season is characterized by stationary hydroclimatic and rainfall properties. At the onset of the dry season, the soil moisture initial condition is a random variable, described by the soil moisture PDF following the final storm of the wet season, which is an improved representation of conditions at the beginning of the dry season, compared to the steady state wet season PDF. The dry season has zero rainfall and a deterministic soil moisture dry down, which proceeds uninterrupted until the following wet season. Consequently, the approach assumes rainfall seasonality is perfectly binary; i.e., a wet season, characterized by statistically stationary rainfall, is followed by a dry season with no significant rainfall at all. This is most appropriate in locations where a pronounced wet season is followed by a dry season during which rainfall is negligible (or where rainfall inputs are so low that the majority of rainfall is intercepted and does not contribute to soil moisture variations over the rooting depth of local vegetation) [*Savenije*, 2004]. If dry season rainfall is negligible, the time-integrated dry season soil moisture PDF can be computed, making it possible to analytically compute the full, annual soil moisture PDF for seasonally dry regions. We note that one of the advantages of the approach here is that it facilitates other stochastic derivations of ecological relevance, for instance, the dry season soil moisture crossing properties.

The model is analogous to the streamflow model of *Müller et al*. [2014], where an existing stochastic streamflow model [*Botter et al*., 2007] was modified by assuming that the dry season streamflow is a function of the water storage in the catchment at the end of the wet season (a stochastic variable), followed by a deterministic, seasonal recession. Here rather than assuming a deterministic form of streamflow recession during the dry season, we prescribe a deterministic loss rate of water to evapotranspiration, allowing stochasticity to arise in the soil water storage at the end of the last wet season storm.

We tested the model at a seasonally dry site monitored within the AmeriFlux network. AmeriFlux sites record micrometeorological and soil moisture data at high temporal resolutions, making them good candidates for comparisons to theory [*Miller et al*., 2007]. We first compare the empirical annual soil moisture PDF to the modeled annual soil moisture PDF. The annual time scale PDF provides a parsimonious model test: it encodes all the features of a seasonal model, and allows all available soil moisture data to be brought to bear on model testing. Following this, the model's prediction of the dry season mean first passage time below a soil moisture threshold is compared with a previously developed expression for this crossing time [*Viola et al*., 2008] and with data. The model developed by *Viola et al*. [2008] provides an interesting contrast in simplifying assumptions: the current model accounts for end of wet season variability and not dry season rainfall, and the model of *Viola et al*. [2008] incorporates dry season rainfall but neglects variability in the dry season soil moisture initial condition.

## 2. Methods

### 2.1. Symbols Used

Throughout this section, $\Gamma(*)$ refers to the gamma function, $\Gamma(*,*)$ to the generalized incomplete gamma function, and $\delta(*)$ to the Dirac delta function [*Abramowitz and Stegun*, 1964]. The probability density function is represented by $p$ and the cumulative density function (CDF) by $P$. Subscripts, in upper case, denote the random variable being described by the PDF or CDF, and the corresponding lower case characters denote the observed value of the random variable. For example, the PDF and the CDF of the soil moisture $S$ at value $s$ are denoted as $p_S(s)$ and $P_S(s)$, respectively.

### 2.2. Seasonally Dry Stochastic Soil Moisture: Wet Season Soil Moisture and the Dry Season Initial Condition

The mass balance for water with constant density within a one-dimensional control volume spanning the active rooting zone depth, Zr [L], is given by:

$$nZ_r\frac{ds}{dt}=R(t)-\mathrm{ET}[s(t)]-\mathrm{LQ}[s(t),t], \tag{1}$$

where $n$ is the porosity, LQ [$L\ T^{-1}$] is the flux of soil moisture leaving the control volume as runoff or deep drainage, and ET [$L\ T^{-1}$] is the flux of soil moisture lost due to evapotranspiration. Following *Porporato et al.* [2004], this model simplifies the dependence of evapotranspiration and drainage dynamics on soil water content by assuming that any water storage in excess of field capacity ($s_1$) is instantly drained. While evapotranspiration is assumed to occur at a prescribed maximum rate $\mathrm{ET}_{max}$ at $s_1$, this rate of loss declines linearly until it goes to zero evapotranspiration at the wilting point ($s_w$):

$$\mathrm{ET}(s)=\begin{cases}0 & : s\in(0,s_w)\\ \mathrm{ET}_{max}\cdot\dfrac{s-s_w}{s_1-s_w} & : s\in[s_w,s_1]\end{cases}. \tag{2}$$

Rainfall, R(t), is modeled on daily time scales as a Poisson process with exponentially distributed depths, making equation (1) a stochastic differential equation.

Under stationary climate conditions, assumed to prevail during the wet season in a seasonally dry climate, the steady state PDF of the nondimensional, wet-season, relative soil moisture $X_w$ (scaled to assume a value of zero at the wilting point and a value of one at field capacity, $x=\frac{s-s_w}{s_1-s_w}$) can be obtained [*Porporato et al.*, 2004]:

$$p_{X_w}(x_w)=\frac{N}{\eta}x_w^{\lambda/\eta-1}e^{-\gamma x_w}\quad \text{for}\quad x_w\in[0,1], \tag{3}$$

The model is characterized by the two nondimensional parameters:

$$\gamma=\frac{w_0}{\alpha}\quad \text{and}\quad \frac{\lambda}{\eta}=\frac{\lambda w_0}{\mathrm{ET}_{max}}, \tag{4}$$

where $w_0[L]=(s_1-s_w)nZ_r$ is the total available water storage, $\lambda$ [$T^{-1}$] is the reciprocal of the mean waiting time between rainfall events, $\alpha$ [$L$] is the mean depth of the rainfall events, $\mathrm{ET}_{max}$ [$L\ T^{-1}$] is the maximum rate of evapotranspiration from the soil, and $N$ is a normalization constant. Due to seasonal changes in temperature, insolation and relative humidity, $\mathrm{ET}_{max}$, is likely to vary between the wet and dry seasons, leading also to different values of $\eta$. We therefore use $\eta_w$ to describe the wet season dynamics and $\eta_d$ to describe the dry season dynamics.

Moisture dynamics during the dry season are assumed to consist of a deterministic dry down due to ongoing evapotranspiration from the soil. The dry season soil moisture initial condition, $X_0$, is treated as a stochastic variable generated by the last significant storm of the wet season. This soil moisture state at the onset of the dry season is the sum of the soil moisture condition that prevailed before the final wet season storm and the soil moisture increment introduced by this storm (this biases $X_0$ toward a more saturated state than prevails during the wet season as a whole). To determine the distribution of this initial condition, we first nondimensionalize the depth of each incoming rainfall event ($R$) by the transformation:

$$R'=\frac{R}{nZ_r}, \tag{5}$$

and then perform the rescaling:

$$H=\frac{R'}{s_1-s_w}=\frac{R}{w_0}, \tag{6}$$

which ensures that the new increment, $H$, has the same nondimensional scale as $Xw$. Based on the assumption that rainfall depths are exponentially distributed, the CDF and PDF of $H$ are given by:

$$P_H(h)=1-e^{-\gamma h} \quad \text{and} \quad p_H(h)=\gamma e^{-\gamma h}. \tag{7}$$

Since the Poisson rainfall process is memoryless, the soil moisture conditions prior to the final wet season storm are described simply by the steady state wet season soil moisture PDF, $p_{X_w}$. The dry season soil moisture initial condition ($X_0$) is the sum of the wet season soil moisture ($X_w$) and the rainfall depth ($H$) random variables:

$$X_0=X_w+H, \tag{8}$$

assuming $X_w+H<1$. There is a finite probability that the final rainfall increment of the wet season will lead to saturated soil conditions, so the initial condition distribution also contains an atom of probability at $X_0=1$. For a final rainfall increment that causes $X_0<1$, the PDF of $X_0$ conditioned on $X_0<1$ is:

$$p_{X_0|X_0<1}(x_0)=\int_0^{x_0} p_{X_w}(x)p_H(x_0-x)dx=\frac{N\gamma}{\lambda}e^{-\gamma x_0}x_0^{\frac{\lambda}{\eta_w}} \quad \text{for} \quad x_0\in(0,1). \tag{9}$$

The atom of probability at $X_0=1$ is given by:

$$\Pr(X_0=1)=\int_0^1 p_{X_w}(x)(1-P_H(1-x))dx=\frac{N}{\lambda}e^{-\gamma}, \tag{10}$$

where $1-P_H(1-x)$ is the probability that the final rainfall increment is greater than $1-x$, leading to saturated conditions at the end of the wet season. The PDF for the soil moisture conditions at the start of the dry season is:

$$p_{X_0}(x_0)=\frac{N\gamma}{\lambda}e^{-\gamma x_0}x_0^{\frac{\lambda}{\eta_w}}+\frac{N}{\lambda}e^{-\gamma}\cdot\delta(x_0-1) \quad \text{for} \quad x_0\in(0,1]. \tag{11}$$

### 2.3. Dry Season Soil Moisture PDF

In addition to previous studies, which derive the probabilistic dynamics of dry season soil moisture at the daily ($X_d^t$) time scale, we derive the PDF for dry season soil moisture at the annual ($X_d$) time scale. $X_d^t$ and $X_d$ represent slightly different ways to quantify soil moisture values. The former is a soil moisture PDF that changes as a function of the number of days ($t$) from the start of the dry season. On the first day ($t=0$), $X_d^t$ is necessarily distributed as $X_0$. In previous studies [*Viola et al*., 2008], $X_d^t$ also varies depending on the likelihood of rainfall occurring. In this study, assuming no significant dry season rainfall, evapotranspirative processes deplete soil water stores as the dry season progresses, biasing $X_d^t$ toward more dry conditions. In contrast, the time-independent dry season soil moisture distribution, $X_d$, represents the time-integrated probability distribution of soil moisture over the course of the dry season. It is therefore independent of time. This distribution combines all dry season soil moisture data into a single, lumped distribution, encapsulating variability from the antecedent wet season and from the temporal evolution of the daily dry season soil moisture.

#### 2.3.1. The Distribution of $X_d^t$

In the absence of dry season rainfall, the dry season soil moisture dynamics given by the soil water balance (equation (1)) are described by a deterministic, exponential dry down:

$$x_d^t=x_0e^{-\eta_d t} \Rightarrow x_0=x_d^t e^{\eta_d t}, \tag{12}$$

where $\eta_d$ is the dry season equivalent of $\eta_w$. Equation (12) specifies a unique dry season soil moisture value for a given initial condition ($x_0$) and a given number of days into the dry season ($t$). This implies that the time-dependent, dry season soil moisture is a derived random variable $\left(X_d^t\right)$ of the dry season initial condition:

$$\begin{aligned} p_{X_d^t}\left(x_d^t\right) &= p_{X_0}\left(x_0\left(x_d^t\right)\right)\frac{dx_0}{dx_d^t} \\ &= e^{\eta_d t}\cdot\left[\frac{N\gamma}{\lambda}\exp\left(-\gamma x_d^t e^{\eta_d t}\right)\left(x_d^t e^{\eta_d t}\right)^{\lambda/\eta_w}+\frac{N}{\lambda}e^{-\gamma}\delta\left(x_d^t-e^{-\eta_d t}\right)\right] \text{ for } x_d^t\in(0,e^{-\eta_d t}]. \end{aligned} \tag{13}$$

The time-dependent moments of $X_d^t$ can be obtained from its moment generating function M(c) [*Ross*, 2009], which is defined as:

$$M(c)=\mathbb{E}[\exp(c\cdot x_0e^{-\eta_d t})]=\frac{N}{\lambda}e^{-\gamma}\cdot\exp(c\cdot e^{-\eta_d t})+\int_0^1 p_{X_0}(x_0)\cdot\exp(c\cdot x_0e^{-\eta_d t})dx_0$$
$$=\frac{N\left[\gamma(\gamma-ce^{-\eta_d t})^{-\frac{\eta_w+\lambda}{\eta_w}}\left[\Gamma\left(\frac{\eta_w+\lambda}{\eta_w}\right)-\Gamma\left(\frac{\eta_w+\lambda}{\eta_w},\gamma-e^{-\eta_d t}c\right)\right]+e^{ce^{-\eta_d t}-\gamma}\right]}{\lambda}. \tag{14}$$

For example, the mean of $X_d^t$ is then calculated from $M(c)$ as:

$$\langle X_d^t\rangle=\frac{dM}{dc}\bigg|_{c=0}=\frac{N(\eta_w+\lambda)\gamma^{-\frac{\eta_w+\lambda}{\eta_w}}\left[\Gamma\left(\frac{\eta_w+\lambda}{\eta_w}\right)-\Gamma\left(\frac{\eta_w+\lambda}{\eta_w},\gamma\right)\right]}{\eta_w\lambda}\cdot e^{-\eta_d t}=\langle X_0\rangle e^{-\eta_d t}, \tag{15}$$

which is exactly the form of the exponential dry down with $x_0=\langle X_0\rangle$. This approach could be extended to calculate higher-order moments of $X_d^t$, such as its variance, or to incorporate temporal variability in the climate parameters. For example, $\eta_d=\eta_d(t)$ can be substituted into equation (14) without affecting the analytical tractability of the moment calculations.

#### 2.3.2. The Distribution of $X_d$

For simplicity, we assume that the duration of the dry season ($t_d$) is constant from year to year. Provided that the available soil storage does not grossly exceed the mean soil moisture increment and that the wet season duration is long compared to the average rainfall interarrival time, the duration of the transient wet-up period at the start of the wet season can be assumed insignificant compared to the duration of the entire wet season, and the distribution of soil moisture during the wet season will be independent of the wet season length (note that both of these conditions would likely be violated in very arid regions, meaning that these models are inappropriate for representing soil moisture dynamics in true deserts). If the assumption of independence is valid, then the annual soil moisture PDF can be calculated as a weighted sum of the dry season and wet season soil moisture PDFs:

$$p_X(x)=\left(1-\frac{t_d}{365}\right)p_{X_w}(x)+\frac{t_d}{365}p_{X_d}(x). \tag{16}$$

To calculate the time-integrated dry season PDF, we first note that the CDF of the dry season soil moisture ($X_d$) conditioned on the value of $X_0$ is:

$$P_{X_d|X_0}(x_d,x_0)=P\{X_d\le x_d|X_0=x_0\}$$
$$=\begin{cases}0 & : x_d\in(0,x_0e^{-\eta_d t_d})\\ \dfrac{\ln\left(\dfrac{x_0}{x_d}\right)}{\eta_d t_d} & : x_d\in[x_0e^{-\eta_d t_d},x_0)\\ 1 & : x_d\in[x_0,1).\end{cases} \tag{17}$$

The logic behind the second line of (17) is the (admittedly obvious) observation that time itself is uniformly distributed over the course of the dry season; that is, each day from the dry season has an equal probability of being selected in a random sample of days from the dry season. This implies that the CDF of time over the dry season is a simple linear function: $P_T(t)=t/t_d\ :\ t\in[0,t_d]$. Since, for a given initial condition, the dry season soil moisture is only a function of time, the distribution of the integrated dry season soil moisture random variable is a derived distribution of the uniform distribution of time. The expression $x_d=x_0e^{-\eta_d t}$ therefore can be solved for $t$ and then substituted into $P_T(t)=t/t_d$ to obtain the CDF of $X_d$ (with an appropriate transformation of the domain) which is the time-integrated form of $X_d^t$. The time-integrated dry season PDF of $X_d$ given $X_0=x_0$ is then:

$$p_{X_d|X_0}(x_d,x_0)=\frac{dP_{X_d|X_0=x_0}}{dx_d}$$
$$=\frac{1}{t_d\,x_d\,\eta_d}\quad\text{for}\quad x_0e^{-\eta_d t_d}\le x_d\le x_0. \tag{18}$$

The unconditional, time-integrated dry season PDF can be found from the conditional distribution of equation (18) by integrating over the distribution of $X_0$:

$$p_{X_d}(x_d)=\int_{X_0} p_{X_d|X_0}(x_d,x_0)\,p_{X_0}(x_0)dx_0$$

$$=\begin{cases}\displaystyle\int_{x_d}^{x_d e^{\eta_d t_d}} p_{X_d|X_0}(x_d,x_0)p_{X_0}(x_0)dx_0 & : x_d\in(0,e^{-\eta_d t_d})\\[2ex] \displaystyle\int_{x_d}^{1} p_{X_d|X_0}(x_d,x_0)p_{X_0}(x_0)dx_0 & : x_d\in[e^{-\eta_d t_d},1]\end{cases} \tag{19}$$

$$=\begin{cases}\dfrac{N\gamma^{-\frac{\lambda}{\eta_w}}\left[\Gamma\left(\dfrac{\eta_w+\lambda}{\eta_w},x_d\gamma\right)-\Gamma\left(\dfrac{\eta_w+\lambda}{\eta_w},x_d\gamma e^{t_d\eta_d}\right)\right]}{t_d\,\eta_d\,\lambda\,x_d} & : x_d\in(0,e^{-\eta_d t_d})\\[3ex] \dfrac{N\gamma^{-\frac{\lambda}{\eta_w}}\left[\Gamma\left(\dfrac{\eta_w+\lambda}{\eta_w},x_d\gamma\right)-\Gamma\left(\dfrac{\eta_w+\lambda}{\eta_w},\gamma\right)\right]}{t_d\eta_d\lambda x_d}+\dfrac{Ne^{-\gamma}}{\lambda\,t_d\,x_d\,\eta_d} & : x_d\in[e^{-\eta_d t_d},1].\end{cases}$$

There are two distinct domains for the distribution $p_{X_d}$: (1) $x_d\in(0,e^{-\eta_d t_d})$, and (2) $x_d\in[e^{-\eta_d t_d},1)$. The only way for the dry season soil moisture to take on a value in the first domain (that is, the only source of probability density in that domain) comes from a dry season soil moisture initial condition that is greater than the value of $x_d$ itself (hence the lower bound), but less than the initial condition value which leads to a dry season soil moisture value of $x_d$ on the very last day of the dry season (hence the upper bound at $x_d e^{\eta_d t_d}$). In the second domain, however, there exist values for $x_0$ that are greater than 1 (which is outside the domain of $p_{X_0}$) which could lead to a dry season soil moisture value $x_d\in[e^{-\eta_d t_d},1]$. Therefore, the upper bound on $x_0$ should be fixed to 1 in this domain, while the lower bound remains the same.

The annual soil moisture PDF is then calculated as the weighted sum of the dry season and wet season soil moisture PDFs, according to equation (16).

### 2.4. Dry Season Mean First Passage Time

As a simplified measure of dry season plant water stress in regions where the growing season and the dry season coincide, *Rodríguez-Iturbe et al*. [2001] and Viola et al. [2008] consider the mean fraction of the dry season that soil moisture is less than some threshold ($s^*$), below which plants become water stressed:

$$\text{Plant water stress} \propto \frac{t_d-\bar{T}_{s^*}}{t_d}. \tag{20}$$

Here $\bar{T}_{s^*}$ is the mean time from the start of the growing (dry) season to reach the water stress threshold $s^*$. Using a number of simplifying assumptions, *Viola et al*. [2008] derive the following approximate expression for $\bar{T}_{s^*}$:

$$\bar{T}_{s^*}=\frac{\min\left[(s_1-s^*)nZ_r,\,t_w\left(\alpha_w\lambda_w-ET_{\max,w}\right)-(s^*-s_w)nZ_r\right]}{ET_{\max,d}-\alpha_d\lambda_d}, \tag{21}$$

where $\alpha_w,\lambda_w$ are wet season rainfall statistics (mean depth and frequency), $t_w$ is the duration of the wet season, $\alpha_d,\lambda_d$ are dry season rainfall statistics, and $ET_{\max,w}, ET_{\max,d}$ are the wet and dry season maximum rates of evapotranspiration. This expression assumes that the soil moisture conditions at the beginning of the growing season are well approximated by the mean wet season soil moisture conditions and that the threshold crossing time is a linear function of this initial condition.

The model derived by *Viola et al*. [2008] accounts for stochasticity in $T_{s^*}$ that results from dry season rainfall processes. The theory presented here yields an analogous expression for $T_{s^*}$ that instead accounts for variability in the dry season initial soil moisture condition. First, we use equation (12) to obtain an expression for $T_{x^*}$ (where $x^*$ is the normalized equivalent of the relative soil moisture threshold, $x^*=\frac{s^*-s_w}{s_1-s_w}$) as a function of the dry season initial condition:

$$x^*=x_0e^{-\eta_d T_{x^*}} \quad \Rightarrow \quad T_{x^*}(x_0)=\frac{\ln\frac{x_0}{x^*}}{\eta_d}. \tag{22}$$

Making the simplifying assumption that the dry season initial condition is greater than the soil moisture stress threshold with high probability $(\text{Prob}[X_0 > x^*] \gg \text{Prob}[X_0 < x^*])$, the mean of $T_{x^*}$ is easily obtained by integrating over the PDF for $X_0$:

$$T_{x^*}=\int_{x^*}^{1} p_{X_0}(x_0)T_{x^*}(x_0)x_0$$

$$=\frac{N}{\eta_d\lambda}\left\{\frac{\gamma\,\eta_w^2\,x^{*\frac{\eta_w+\lambda}{\eta_w}}\,A\left(\frac{\lambda}{\eta_w}+1,\frac{\lambda}{\eta_w}+1;\frac{\lambda}{\eta_w}+2,\frac{\lambda}{\eta_w}+2;-x^*\gamma\right)}{(\eta_w+\lambda)^2}-\frac{\gamma\,\eta_w^2\,A\left(\frac{\lambda}{\eta_w}+1,\frac{\lambda}{\eta_w}+1;\frac{\lambda}{\eta_w}+2,\frac{\lambda}{\eta_w}+2;-\gamma\right)}{(\eta_w+\lambda)^2}+\right.$$

$$\left.\ln x^*\left[\gamma\,\mathrm{E}\left(-\frac{\lambda}{\eta_w},\gamma\right)-\gamma^{-\frac{\lambda}{\eta_w}}\Gamma\left(\frac{\eta_w+\lambda}{\eta_w}\right)-e^{-\gamma}\right]\right\}, \tag{23}$$

where $E(n,z)$ is the exponential integral function $E_n(z)=\int_1^{\infty}\frac{e^{-zt}}{t^n}dt$ and A is a version of the generalized hypergeometric function [*Abramowitz and Stegun*, 1964]:

$$A(a_1,a_2;b_1,b_2;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}, \quad \text{with}$$
$$(a)_0=1 \quad \text{and} \quad (a)_n=a(a+1)(a+2)\ldots(a+n-1), \quad n\geq 1 \tag{24}$$

## 2.5. Case Study

### 2.5.1. Site

A case study from the AmeriFlux station at Tonzi Ranch is presented to empirically test the annual soil moisture distribution and mean first crossing time models. Tonzi Ranch is an oak savanna woodland located in the foothills of the Sierra Nevada near Ione, California. The climate is Mediterranean, characterized by wet, cool winters and hot, dry summers [*Ma et al.*, 2007; *Baldocchi et al.*, 2010]. Twelve years of soil moisture data (2001–2013), collected at depths of 5, 20, and 50 cm using Theta Probe model ML2-X impedance sensors (Delta-T Devices) [*Miller et al.*, 2007], are analyzed for this study.

### 2.5.2. Model Parameterization

The eight model parameters ($t_d$, $\mathrm{ET}_{\max}$, $\alpha$, $\lambda$, $s_w$, $s_1$, $Z_r$, $n$) were computed from the AmeriFlux data sets. One of the challenges in applying our model is that the assumption of binary rainfall seasonality is an approximation at best, and the model user must make decisions about how to pragmatically separate the wet and dry periods. Numerous rubrics could feasibly be used to differentiate these seasons, leading to variable results. Here we present results based on two distinct partitioning rubrics, to explore the sensitivity of the results to reasonable choices.

For Rubric 1, we simply examine the spring months from March through the end of May and choose the latest well-defined (change in sign of the first derivative) soil moisture peak that is greater than a threshold, chosen here to be $x=0.6$. For Rubric 2, we first extract each well-defined soil moisture peak during the spring period as potential start days for the dry season. We then calculate the mean of all the soil moisture local minima from 1 January to each potential dry season starting peak. The chosen peak is the final peak of the wet season greater than the mean of the preceding soil moisture minima. This ensures that the selection of the initial condition is strongly correlated with wet season conditions. In both cases, the end of the dry season is chosen as the minimum soil moisture value between the start of the dry season and the start of the next calendar year. Figure 1 presents a plot of the seasonally partitioned soil moisture time series corresponding to each rubric. The dry season length ($t_d$) is calculated by subtracting the median wet season length from 365. Each extracted dry season is used to calculate the first passage time below soil moisture thresholds ranging from 1% to 50% of the soil moisture wilting point.

We estimated $\mathrm{ET}_{\max}$ using the Priestly-Taylor equation:

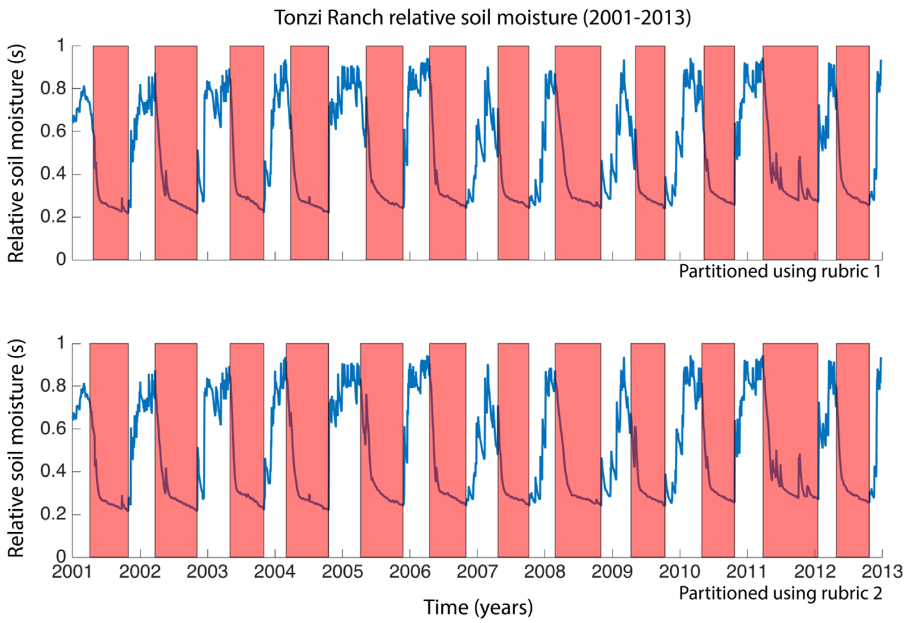


**Figure 1.** Relative soil moisture time series data from Tonzi Ranch. Extracted dry seasons are denoted with shaded rectangles. The top row uses seasonality partitioning Rubric 1, and the bottom row uses Rubric 2.

$$\mathrm{ET}_{\max}=1.26\,\frac{e_s'}{(e_x'+g)\,L}(R_n-G), \tag{25}$$

where $g$ is the psychrometric constant, $L$ is the latent heat of vaporization of water, $G$ the ground heat flux, $R_n$ the net radiation, and $e\prime_s$ is the derivative of the saturation vapor pressure (calculated using the Clausius-Clapeyron equation) with respect to temperature. The value of $\mathrm{ET}_{\max}$ was computed separately for each season based on the seasonal daily mean value of $\mathrm{ET}_{\max}$. The mean rainfall depth and mean waiting time between rainfall events (for wet and dry seasons), $\alpha$ and $\lambda$, were derived by aggregating 30 min AmeriFlux rain gauge data to the daily time scale. Due to the fact that the duration of Pacific coastal storm systems is typically longer than 1 day, we treated multiday storm events as single events. Similarly to previous sensitivity analyses using stochastic streamflow models [*Müller et al*., 2014], however, we find that this deviation from the exact specification of rainfall as a Poisson process does not significantly degrade model performance.

The soil depth ($Z_r$) and porosity ($n$) were obtained from the Ameriflux biological data. Although soil textural data are available for the site, using a pedotransfer function [e.g., *Saxton and Rawls*, 2006] to estimate the model parameters is problematic for the *Porporato et al*. [2004] model, since the simplifications made to the drainage and evaporation processes mean that $s_1$ and $s_w$ in the model do not map precisely to conventional definitions of field capacity and wilting point. We therefore calibrate these parameters, recognizing that these values primarily affect the validity of the wet season PDF (i.e., the underlying *Porporato et al*. [2004] model).

**Table 1.** Tonzi Ranch—Site Characteristics Calculated From AmeriFlux Data

| Parameter | Rubric 1 | Rubric 2 |
|---|---|---|
| $t_d$ (days) | 202 | 212 |
| $\mathrm{ET}_{\max}$ wet season (dry season) (mm/d) | 2.01 (5.54) | 1.88 (5.50) |
| $\alpha$ wet season (dry season) (mm) | 24.45 (5.52) | 24.63 (6.67) |
| $\lambda$ wet season (dry season) (day$^{-1}$) | 0.14 (0.04) | 0.14 (0.04) |
| $s_w$ | 0.26 | 0.26 |
| $s_1$ | 0.82 | 0.82 |
| $Z_r$ (mm) | 600 | 600 |
| $n$ | 0.45 | 0.45 |

Bulk volumetric soil water content is calculated by zonally averaging the soil moisture measurements over the depth of the soil column. For this type of averaging, each depth is assigned the soil moisture value of the nearest measured value, then the standard integrated average of the resulting profile is computed. Table 1 summarizes model parameters.

### 2.5.3. Model Evaluation

To evaluate the performance of our model, we used the Nash-Sutcliffe efficiency (*NSE*) applied to the soil moisture quantiles:

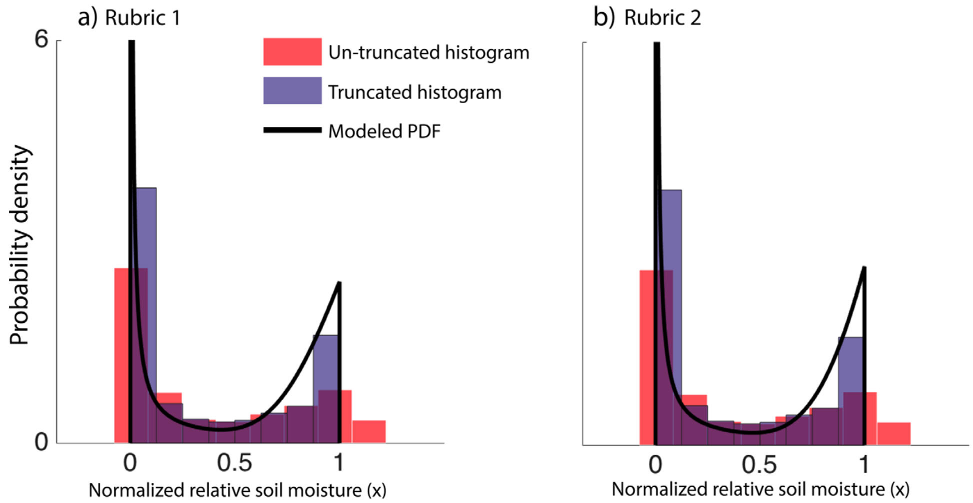


**Figure 2.** Integrated annual soil moisture PDFs compared to AmeriFlux soil moisture data, partitioned using (a) Rubric 1 and (b) Rubric 2.

$$NSE = 1 - \frac{\sum_{i=1}^{364} (\hat{q}_i - q_i)^2}{\sum_{j=1}^{364} \left( q_j - \frac{1}{364} \sum_{k=1}^{364} q_k \right)^2}, \tag{26}$$

where $\hat{q}_i$ and $q_i$ are the empirical and theoretical relative soil moisture values associated with quantile $i$ of the annual soil moisture PDF. The *NSE* has been used extensively for the assessment of hydrologic models [*Nash and Sutcliffe*, 1970; *Castellarin et al.*, 2004; *Müller et al.*, 2014].

## 3. Results and Discussion

The case study shown in Figure 2 supports the applicability of the model for capturing the annual PDF of soil moisture in seasonally dry regions. Although the domain of the analytical PDF is supported on $x \in [0, 1]$, the raw data take on values outside of this range. Figure 2 therefore shows the raw empirical histogram and a truncated histogram, the latter computed by setting all soil moisture values above field capacity ($x = 1$) to field capacity and all values below the wilting point ($x = 0$) to the wilting point. The two seasonal partitioning rubrics (Figures 2a and 2b) yield similar results at the Tonzi Ranch site. Using Rubric 1 and untruncated soil moisture data, the model *NSE* is 0.85, indicating that even the simple bounded model yields a good fit to field measurements. For the truncated quantiles, the *NSE* for the analytical model is 0.86. Using Rubric 2, the model *NSE* is 0.89 against the untruncated soil moisture data and 0.90 against the truncated soil moisture data.

The computed mean first passage times for each partitioning rubric are plotted in Figure 3. There is strong agreement between the mean first passage time for the current model using both rubrics, suggesting that dry season rainfall plays an insignificant role at Tonzi Ranch in determining the time for soil moisture to drop below the presented range of soil moisture thresholds. We also found that even for the lowest threshold, set to 1% of the soil moisture wilting point, all dry seasons were long enough for soil moisture levels to drop below the threshold. This implies that variability in the dry season length at this site is not an issue when considering first passage times, as truncation (due to the beginning of the next wet season) of the seasonal dry down (above ecological stress thresholds, for instance) is unlikely. The model of *Viola et al.* [2008] works well for higher soil moisture thresholds, but underestimates the mean crossing time for lower thresholds. These differences likely arise from the fact that, in order to obtain an approximate analytical expression for $\bar{T}_{s^*}$, *Viola et al.* [2008] both linearize the dry season soil moisture dynamics and use a simplified expression for the soil moisture conditions at the start of the dry season. Consequently, the new model, which includes a more accurate description of the dry season initial conditions and does not linearize the

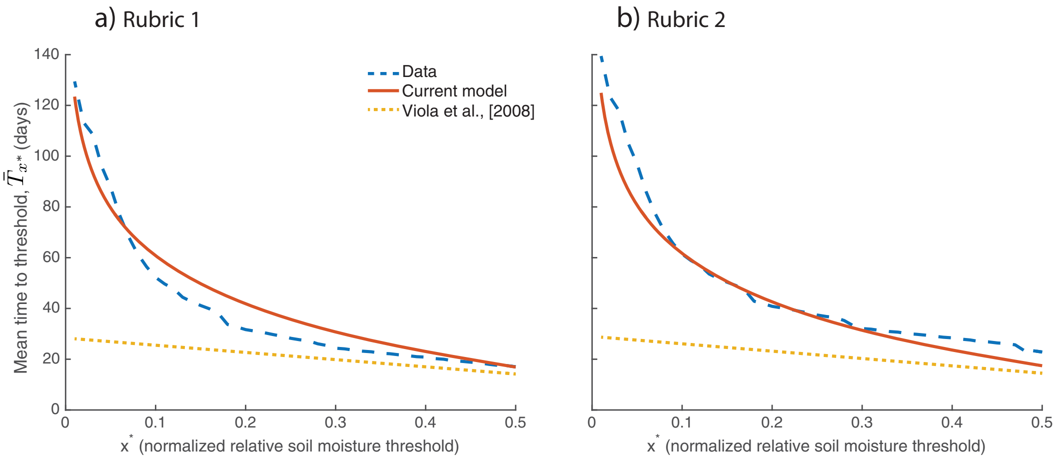


**Figure 3.** Mean time from the beginning of the dry season to reach a range of soil moisture thresholds ($x^*$) using Ameriflux data (blue dashed line), the current model (solid line), and the model developed by *Viola et al.* [2008] (finely dashed line).

functional form of the soil moisture dry down, outperforms the model presented by *Viola et al.* [2008]. We suspect, however, that at sites with more significant dry season rainfall contributions, the model of *Viola et al.* [2008] would become increasingly more appropriate.

Clearly the model presented here could be further elaborated, for example, through use of a more complete soil moisture model, such as *Laio et al.* [2002]. More realistic loss functions can be incorporated into the soil moisture model without significantly altering the logic of the approaches illustrated here. However, the use of a more complex loss function leads to considerably more complex algebra to manage the conditionality associated with multipart piecewise functions, variable initial conditions, and a finite dry season length. Here we elected to base the analysis on the simpler *Porporato et al.* [2004] model to ensure that the logic of the approach was not obscured.

The model also makes the seemingly inconsistent assumption that the dry season can have a variable start date, but that season lengths are fixed. To resolve this, the stochastic model should not be treated as a one-to-one mapping of the soil moisture time series into a probabilistic domain, but rather as a model developed in the probabilistic domain. The parameterization represents the deliberate decision to capture the important source of variation imposed by conditions at the end of the wet season, while retaining analytical tractability. The model then assumes that this form of variability is more important than creating a perfect mapping between the time series and probability domains.

Other sources of stochasticity, such as the dry season maximum evapotranspiration rates, could also be incorporated. Interannual variations in $ET_{max}$, however, are subordinate to interannual variations in rainfall when driving hydrological processes [see, for example, *Milly and Dunne*, 2002], while variability in the length of the dry season could only be expected to influence the soil moisture PDF when the time scales of dry down approach the mean dry season length. Thus, the level of complexity used here sufficiently captures key stochastic drivers of soil moisture dynamics in seasonally dry systems.

## 4. Conclusion

This work presents an analytical model to compute the PDF of a bounded random variable, soil moisture, in climates with two distinct seasons. The formulation is used to derive a simple analytical expression for the dry season mean time to reach a threshold of water stress $s^*$. The presented model and an existing model are tested and compared using soil moisture data from the Tonzi Ranch Ameriflux site. The case study demonstrates that the current model performs well, despite simplifications in the underlying evapotranspiration and drainage dynamics, and may be particularly valuable in regions such as California characterized by pronounced seasonality in rainfall, and large fluctuations around mean wet season soil moisture.

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